Optimal design of multi-layer thermal protection of variable thickness

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Abstract

Purpose – The presented paper aims to consider algorithm for optimal design of multilayer thermal insulation.

Design/methodology/approach – Developed algorithm is based on a sequential quadratic programming method.

Findings – 2D mathematical model of heat transfer in thermal protection was considered in frame of thermal design of spacecraft. The sensitivity functions were used to estimate the Jacobean of the object functions.

Research limitations/implications – Design of distributed parameter systems and shape optimization may be thought of as geometrical inverse problems, in which the positions of free boundaries are determined along with the spatial variables. In such problems, the missing data (i.e. the position of boundaries) are compensated for by the presence of the so-called inverse problem additional conditions. In the case under consideration, such conditions are constrains on the temperature values at the discrete points of the system.

Practical implications – Results are presented how to apply the algorithm suggested for solving a practical problem – thickness sampling for a thermal protection system of advanced solar probe.

Originality/value – The procedure proposed in the paper to solve a design problem is based on the method of quadratic approximation of the initial problem statement as a Lagrange formulation. This has allowed to construct a rather universal algorithm applicable without modification for solving a wide range of thermal design problems.

Keywords Multi-layer thermal protection, Shape design, Solar probe

Paper type Research paper

Nomenclature

Δ_l = Layer thickness vector
ρ_l = Density vector
L = Number of layers in system
T_l = Temperature distribution at the l-th layer
r = Radial coordinate
θ = Polar angle
τ = Time
R_l = Radial coordinate at the boundary of layer
C_1 = Heat capacity
λ_1 = Thermal conductivity
A_l = Contact thermal resistance

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β = Parameter characterized a boundary condition
α = Parameter characterized a boundary condition
q = Heat flux
N = Number of layers which thickness is to be determined
T_m = Temperature constraint at the point with coordinate R_m
N_τ = Number of steps by time
N_θ = Number of steps by polar angle
M = Number of constraints
φ_{ik} = Basis functions
J = Minimized functional
Λ = Lagrangian function
$\bar{\Psi}^T$ = Vector of Lagrangian multipliers
$\bar{T}^T$ = Constraints vector
A = Jacobian matrix of constraints
H = Hessian matrix
Φ = Objective function
$\bar{d}$ = Nonoptimal value of desired vector
$\Delta\bar{d}$ = Search direction
$\bar{p}$ = Gradient of Lagrangian function
γ = Descent step

Introduction

Investigation of the close vicinity of the Sun is one of the major problems of astrophysics, significant for understanding of fundamental physical processes in the solar atmosphere responsible for the magnetic activity, the heating of the corona and energetic-particle acceleration. Despite the recent great achievements in the Sun exploration, many questions, concerning its nature, remain unanswered. A mission to the innermost regions of the heliosphere, providing a combination of in-situ and remote-sensing observations, would represent the next step forward in the exploration of the Sun and allow determining the structure and dynamics of the solar atmosphere, the corona heating mechanisms and the origin and evolution of the solar wind. The acquisition of the in-situ measurements together with the high-resolution imaging and spectroscopy from a near-Sun and out-of-ecliptic perspective made from series of heliocentric orbits with gradually decreasing perihelion distances is the primary scientific objective of the solar probe that has been developed in the framework of "Interhelio-Zond" project.

The extreme environments to be encountered by the spacecraft during perihelion passes require a sophisticated thermal design that can accommodate a wide range of heat loads. To ensure required operational temperature for the instruments and components inside the spacecraft bus solar probe utilizes a sun shield, protecting it from the direct exposure to the intense solar flux. The main problem in the developing of thermal-loaded structure members, unit members or systems is how to sample optimal design parameters of an object under consideration. As a rule in complex problems the optimal parameters search is realized stage-by-stage based on decomposition of a design problem to a few levels with different detailed elaboration of mathematical models and search operations. The topic of present paper is concerned with development of thermal protection of advanced solar probes. In the case of cone or spherical thermal protection, it is very important to provide the weight effectiveness of thermal protection using a variable thickness of it, dependent on external thermal loading in different points of surface.

In the general case, a thermal design problem can be presented as following (Alifanov, 1978): it is necessary to find a vector of design parameters of a system $\bar{p}$ from some domain P to minimize the object functional $J(\bar{p}, \bar{T})$. The total mass of system, the cost of development and testing of thermal insulation systems and others can be used as a minimized functional (design criterion, optimization criterion, etc.). The domain of feasible solutions is determined by technical and physical constraints in forms of equalities $g_i(\cdot),\ i = 1, 2, \ldots, n$ and inequalities $s_j(\bar{T}) \leq 0,\ j = 1, 2, \ldots, m$. These constraints usually depend on the characteristics of the system's state (temperature, heat fluxes, mass velocity of ablation, concentrations, etc.): $\bar{T}(\bar{x}, \tau) = \{T_k(\bar{x}, \tau)\}_1^K$, where K is determined by the form of the used mathematical model, $\bar{x}$ – spatial coordinate, τ – time. So I have the following formalized problem:

$$\min_{\bar{p} \in P} J(\bar{p}, \bar{T}), \tag{1}$$

$$P = \left\{ \bar{p} \in P \middle| g_i\left(\bar{T}(\bar{x}, \tau), \bar{z}(\bar{T}, \bar{x}, \tau)\right) = 0, i = 1, 2, \ldots, n, S_j\left(\bar{T}(\bar{x}, \tau), \bar{z}(\bar{T}, \bar{x}, \tau)\right) \right.$$
$$\left. \leq 0, j = 1, 2, \ldots, m \right\} \tag{2}$$

$$\bar{T}(\bar{x}, \tau) = L\left(\bar{T}(\bar{x}, \tau), \bar{p}, \bar{x}, \tau, \bar{z}(\bar{T}, \bar{x}, \tau)\right) \tag{3}$$

where $\bar{z}$ – vector of known characteristics (parameters of a mathematical model) of the system under consideration, L – nonlinear operator (mathematical model of heat transfer in the considered system).

The optimal design problem solving is executed with incomplete initial data; therefore, the main approach to determining the design parameter value is a search procedure with the increasing volume of used information and with a constant-increasing degree of detailed elaboration of a mathematical model from stage to stage. A mathematical model of the developed system $L(\ldots)$ in equation (3) and a minimizing object functional J in equation (1) are the basis of optimal thermal design. The model connects the desired parameters (or control functions), heat loads (for example, external and internal heat fluxes) and properties of systems, which are "causes" from the view point of direct problem statement, with characteristics determined by states of system ("effects" – for example, temperature). So, if I follow the conception of cause–effect relationship, a thermal design problem can be considered as inverse heat transfer problem in the extreme statement: using known conditions determined by a feasible thermal state of the object $\bar{T}_P$ in equation (2) (domain of effect), to find the wanted causal characteristics $\bar{p}$, which will satisfy these constraints and minimize criterion J in equation (1).

From the above analysis of thermal design problems statement, one sees that all such problems are inverse problems of mathematical physics from a view point of cause–effect relations in the system considered. One of the most promising directions in solving the inverse problems is to reduce them to an extreme formulation and apply a numerical method of the optimization theory.

All software for calculations in optimal design problems is divided in two parts providing different subject orientation of the solving process, namely:

- a "searching" part, which includes algorithms providing search operations based on a minimization of the properly sampled optimization criteria (algorithms are the object of the analysis in the present paper); and
- a "simulating" part for a simulation of the analyzed system operation process. It includes a heat transfer mathematical models, which connect design parameters with input data (cause–effect relationship set up in the system) based on a primary presentation of the real process in the form of mathematical model.

Determination of the number of layers, types of materials and thickness of layers for multi-layer thermal insulation of minimal mass is one of the most traditional thermal design problems (Alifanov, 1978).

Using an arbitrary mathematical model of the designed system, usually a design problem can be simplified supposing that the optimal layer-thickness in every considered point can be determined independently for each other by a criterion of minimal local mass of the system in a given point (Alifanov, 1978), using one-dimensional heat transfer model.

An optimization problem can be solved if I take into account the constraints on the system statement characteristics (temperature on the boundaries of layer, stressed state of the materials, costs, etc.) through a direct search method both in variation of the number of layers and the materials used. The layer thickness and density vector (Δ_l, $l = 1, 2, \ldots, L$, ρ_l, $l = 1, 2, \ldots, L$) is determined for every such variant to minimize the local mass in the given point r and to satisfy the constraints requirements (Alifanov, 1978).

It should be noted that a solution based on the minimal local mass criterion at specific points of surface can satisfy a developer only in the case when heat loads change not very significant from one point r to another. In the general case, desired vectors $\bar{\Delta}$ and $\bar{\rho}$ (optimal for every rate points r) will not be compatible with each other from technological point of view; therefore, it can be possible that practical realization of a desired $\bar{\Delta}$ as a function of coordinate will be not reasonable because of technological difficulties. So in this case, it is necessary to find compromise solution, which is near to minimized mass solution and satisfies the cost-technology conditions. It can be done, for example, by changing a minimized functional equation (1) as suggested by Alifanov (1978).

The algorithms for similar problem solving were considered in Mikhailov (1980), Meric (1986) and Bushuev and Gorskii (1991). However, in all these publications, the authors used a penalty function method for solving the corresponding optimization problem proposed in early 1960s. The algorithms tell from each other only by technique in calculating the gradients of a minimizing functional (the adjoint variables, sensitivity functions, etc.). But, the method of penalty functions, in spite of its apparent simplicity, has one a very distinct defect: it is necessary to sample a suitable weight functions for every new type of the thermal insulation (or for considerable changing in the properties of materials used). And, it is very difficult to obtain reliable solution of such optimization problem using penalty function method. Difficulties arise if an unsuitable value of the penalty parameter is chosen. If penalty parameter is too small, the region in which iterates converge to solution may be very small. On the other hand, the problem will be ill-conditioned if penalty parameter is too large. Some comparative metrics (just for constant thickness of thermal protection over the spacecraft surface) were presented in Nenarokomov (1997), where sequential quadratic programming (SQP) approach has been compared with penalty function method developed in Mikhailov (1980). Computing times of SQP approaches were about 20 per cent less than traditional penalty functions, but the main preference of SQP approach is that in case of penalty functions method for new thermal protection set, it is necessary to execute three-four preliminary total calculations to find optimal values of penalty functions. In the case of SQP method, it is not necessary.

The proposed approach to an optimal design problem is based on a SQP algorithm with quasi-Newton approximation to the Hessian of Lagrangian function corresponding to an optimization criterion (Gill *et al.*, 1981). This allows building a rather universal algorithm applicable without modifications to solve a wide range of problems of thermal design.

Numerical methods

The optimal design problem of multi-layer thermal insulation is considered in the assumption that a system consists of L layers of different materials with variable thickness in the angular direction $\Delta_l,\ l = 1, 2, \dots, L$ and density $\rho_l,\ l = 1, 2, \dots, L$. Also, a heat transfer process in the system is supposed to be two-dimensional by the spatial coordinate and a transient temperature distribution at the l-th layer $T_l(r, \theta, \tau),\ l = 1, 2, \dots, L$, where τ is time and is covered by the quasi-linear heat conduction equations. The coefficients of parabolic type equations $C_l,\ \lambda_l,\ l = 1, 2, \dots, L$ are functions of temperature. There is a contact heat transfer between the layers characterized by contact thermal resistances $A_l,\ l = 1, 2, \dots, L$, which are also function of temperature. There may be the boundary conditions of the first, second or third kind at both sides of the system considered. In such case, the heat transfer in the considered system is covered by the following set of differential equations:

$$C_l(T)\frac{\partial T_l}{\partial \tau} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda(T)\frac{\partial T_l}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\lambda(T)\sin\theta\frac{\partial T_l}{\partial \theta}\right)$$
$$r \in (R_{l-1}, R_l), \quad \theta \in (0, \theta_0), \quad l = 1, 2, \dots, L, \quad \tau \in (\tau_{\min}, \tau_{\max}] \tag{4}$$

$$T_l(r, \theta, \tau_{\min}) = T_{0l},$$
$$l = 1, 2, \dots, L \tag{5}$$

$$-\beta_1\lambda_1(T_1(R_0, \theta . \tau))\frac{\partial T_1(R_0, \theta, \tau)}{\partial r} + \alpha_1 T_1(R_0, \theta, \tau) = q_1(\theta, \tau) \tag{6}$$

$$-\beta_2\lambda_L(T_L(R_L, \theta, \tau))\frac{\partial T_L(R_L, \theta, \tau)}{\partial r} + \alpha_2 T_L(R_L, \theta, \tau) = q_2(\theta, \tau), \tag{7}$$

$$\lambda_l(T_l(R_l, \theta, \tau))\frac{\partial T_l(R_l, \theta, \tau)}{\partial r} = \lambda_{l+1}(T_{l+1}(R_l, \theta, \tau))\frac{\partial T_{l+1}(R_l, \theta, \tau)}{\partial r},$$
$$l = 1, 2, \dots, L-1, \quad \tau \in (\tau_{\min}, \tau_{\max}], \tag{8}$$

$$-\lambda_l(T_l(R_l, \theta, \tau))A_l(T_l(R_l, \theta, \tau))\frac{\partial T(R_l, \theta, \tau)_l}{\partial r} = T_l(R_l, \theta, \tau) - T_{l+1}(R_l, \theta, \tau), \tag{9}$$

In the general case only, the thickness of l_k-th layers $k = 1, 2, \dots, N,\ N \leq L$ is desired. The thickness of the rest layers can be assigned proceeding from technological and strength considerations.

The criterion for estimation of the unknown vector is, therefore, defined by the local mass functional:

$$J(\bar{\Delta}) = 2\pi \sum_{k=1}^{N} \sum_{l=1}^{L} \int_{0}^{\theta_0} R_{lk}(\theta)\rho_{lk}\Delta_{lk} d\theta \tag{10}$$

where the best estimates will minimize the functional equation (10) and satisfy the following conditions:

$$\Delta_{lk} \geq 0, k = 1, 2, \ldots, N \tag{11}$$

and also the temperature constraints in the separate points of a system will be taken into account:

$$T(R_m, \theta, \tau) \leq T_m, \quad m = 1, 2, \ldots, M, \tag{12}$$

where T_m – is constraints for temperatures at the point with coordinate $r = R_m$. Usually, such points are at boundaries of layers $(T(R_l, \theta, \tau), \; l = 1, 2, \ldots, L)$.

For numerical solution, it is necessary to approximate direct problem equations (4)-(9) on the finite difference grid with N_τ steps by time. Therefore, the constraints in equation (12) can be represented as:

$$T(R_m, \theta_n, \tau_j) \leq T_m, \quad j = 1, 2, \ldots, N_\tau, \quad n = 1, 2, \ldots, N_\theta, \quad m = 1, 2, \ldots M \tag{13}$$

assuming that the grid is thick enough and the fulfilment of condition in equation (13) practically guarantees the fulfilment of condition in equation (12).

It should be also noted that based on a warming-up nature of heat transfer process in a thermal insulation system, in majority of practical cases, the temperatures on the line R_m can touch the constraints T_m only at one time $\tau^* \cong \tau_{j^*}$ and in one point $\theta^* \cong \theta_{n^*}$; therefore, only one constraint from equation (13) can be an active constraint, namely:

$$T(R_m, \theta_{n^*}, \tau_{j^*}) \cong T_m, \quad m = 1, 2, \ldots, M \tag{14}$$

Therefore, further for simplification of formulas, I will use a following notation:

$$T(R_m, \theta_{n^*}, \tau_{j^*}) \leq T_m, \quad m = 1, 2, \ldots, M$$

A question of sampling strategy of a set of active constraints will be considered below.

The unknown parameters and the functional gradient of the considered problem depend on angular coordinate. In this case, the expression for gradient may be obtained representing the desired relations in parametric form.

A universal form of parameterisation is:

$$\Delta_{lk}(\theta) = \sum_{i=1}^{N_k} d_{ik}\phi_{ik}(\theta), \; k = 1, 2, \ldots, N \tag{15}$$

where $d_{ik}, i = 1, 2, \ldots, N_k, k = 1, 2, \ldots, N$ are unknown parameters, $\varphi_{ik}(\theta) i = 1, 2, \ldots, N_k, k = 1, 2, \ldots, N$ is a combination of basis functions. Next, instead of equation (11), I can consider the vector:

$$\bar{d}^T = \{d_{11}, d_{21}, \ldots, d_{N_1 1}, d_{12}, \ldots, d_{N_N N}\} \tag{15a}$$

The positiveness of $\Delta_{lk}(\theta)$ is provided using the technique developed at Artyukhin *et al.* (1993), based on approach analyzed at De Bor (1978). And, the minimized functional may be written in the form:

$$J(\bar{d}) = \bar{d}^T \bar{F}, \tag{15b}$$

where $$\bar{F} = \{f_{ik}\}, \; f_{ik} = 2\pi \int_0^{\theta_0} R_{lk}(\theta) \rho_{lk} \phi_{ik}(\theta) d\theta$$

Thus, the optimal design problem can be reduced to linear functional equation (10) minimization with nonlinear constraints in equations (13), (4 to 9) and constraint equation (11) for the estimated vector. It is known that the solution of such a problem $\bar{d}^*$ is the minimum of the corresponding Lagrangian functions (Gill *et al.*, 1981):

$$\Lambda(\bar{d}, \bar{\Psi}) = J(\bar{d}) - \bar{\Psi}^T \bar{T}, \tag{16}$$

where $\bar{T}^T = \{d_{11}, d_{12}, \ldots, d_{N_k N}, T(R_1, \theta_1, \tau_1) - T_1, \ldots, T(R_M, \theta_{N_\theta}, \tau_{N_\tau}) - T_m\}$ is constraints vector, $\bar{\Psi}^T = \{\Psi_{111}, \ldots, \Psi_{mnj}, \ldots, \Psi_{MN_\theta N_\tau}\}$ is the vector of Lagrange multipliers.

The estimated $\bar{d}^*$ is belonged to a vector subspace, which is orthogonal to subspace of active constraints gradients and satisfies the following conditions (Gill *et al.*, 1981):

- $T(R_m, \theta_n, \tau_j) - T_m \cong 0, \; n = 1, 2, \ldots, N_\theta, \; j = 1, 2, \ldots, N_\tau,$ $m = 1, 2, \ldots, M$ and $T(R_m, \theta_{n^*}, \tau_{j^*}) - T_m = 0,$
- $grad J_d(\bar{d}^*) = \bar{\rho} = A(\bar{d})^T \bar{\Psi}^*,$

where $A(\bar{d}^*)$ is the Jacobian matrix of constraints evaluated at $\bar{d}^*$.

- $\Psi_m^* \geq 0, \quad \Psi_m^* = \Psi_{mn_m^* j_m^*}, \quad m = 1, 2, \ldots, M,$
- $Q(\bar{d}^*)^T W(\bar{d}^*, \bar{\Psi}^*) Q(\bar{d}^*)$ is a positive definite matrix,

where $W(\bar{d}^*, \bar{\Psi}^*)$ – Hessian matrix of Lagrangian function and columns of $Q(\bar{d}^*)$ compose a basis for the null space which is orthogonal to space of rows $A(\bar{d}^*)$.

Therefore, $\bar{d}^*$ can be defined as the solution of a linearly constrained subproblem, whose objective function Φ is related to the Lagrangian function and whose linear constraints are chosen so that a minimization occurs only within the desired subspace. The class of projected Lagrangian methods includes algorithms that contain a sequence of linearly constrained subproblems based on Lagrangian function. And, therefore, the function Φ will include estimates of the Lagrange multipliers ψ.

Let the set of active constraints:

$$T(R_m,\ \theta_{n^*},\tau_{j^*}) \cong T_m, \quad m = 1,\ 2,\ \ldots,\ M \tag{17}$$

is found.

And, suppose also that $\Delta\bar{d}$ is the difference:

$$\Delta\bar{d} = \bar{d}^* - \bar{d}, \tag{18}$$

where $\bar{d}$ is arbitrary nonoptimal value of desired vector or:

$$\mathrm{T}\left(\bar{d}^*, R_m,\ \theta_{nm^*},\ \tau_{jm^*}\right) = \mathrm{T}\left(\bar{d} + \Delta\bar{d},\ R_m, \theta_{nm^*},\ \tau_{jm^*}\right) = \mathrm{T}_m, \quad m = 1,\ 2,\ \ldots,\ M. \tag{19}$$

To calculate the next approximation of $\tilde{\bar{d}}^*$ the Taylor's series are used:

$$T\left(\tilde{\bar{d}}^*, R_m,\ \theta_{nm^*}, \tau_{jm^*}\right) - T_m = T\left(\bar{d}, R_m,\ \theta_{nm^*}, \tau_{jm^*}\right) + A(\bar{d})\Delta\bar{d} + 0\left(\|\Delta\bar{d}\|^2\right) \tag{20}$$

from which I get a set of linear constraints:

$$A(\bar{d})\Delta\bar{d} = -T\left(\bar{d},\ R_m,\ \theta_{nm^*}, \tau_{jm^*}\right) \tag{21}$$

And, therefore, $\tilde{\bar{d}}^*$ is the element of null space of linear approximation of constraints at $\bar{d}$ vectors. Therefore, the initial problem was transformed to a sequence of problems:

$$\min\left\{\bar{p}^T\Delta\bar{d} + \Delta\bar{d}^T H\Delta\bar{d}\right\} \tag{22}$$

$$A(\bar{d})\Delta\bar{d} = -T\left(\bar{d},\ R_m,\ \theta_{nm^*}, \tau_{jm^*}\right) \tag{23}$$

It is reasonable to sample Φ of such a simple form as possible. Most suitable are quadratic functions. So, the problem considered becomes a well-known quadratic programming problem. In this case, the approximation of search direction $\Delta\bar{d}$ is considered as an argument, and the problem is reduced to:

$$\min\left\{\bar{p}^T\Delta\bar{d} + \Delta\bar{d}^T H\Delta\bar{d}\right\} \tag{24}$$

$$A(\bar{d})\Delta\bar{d} = -T\left(\bar{d},\ R_m,\ \theta_{nm^*}, \tau_{jm^*}\right) \tag{25}$$

where $\bar{p}$ is the gradient of Lagrangian function at $\bar{d}$, namely, $\bar{p} = A(\bar{d})^T\bar{\Psi}$, where $\bar{\Psi}$ Lagrangian multiplier vector on a current iteration and the matrix H is a positive-definite quasi-Newton approximation to Hessian of Lagrangian functions.

The explicit method for solving the problem in equations (24)-(25) exists in Gill *et al.* (1981). The vector $\Delta\bar{d}$ is presented as:

$$\Delta\bar{d} = S\bar{d}_s + Z\bar{d}_z, \tag{26}$$

where matrix S columns are the elements of matrix $A(\bar{d})$ rows' subspace, and matrix Z columns compose a basis for a vector subspace which is orthogonal to the subspace of $A(\bar{d})$ rows. Matrixes Z and S can be defined from the RV factorization of $A(\bar{d})$:

$$A(\bar{d}) \cdot V = R, \tag{27}$$

where R is a lower triangular $m \times m$ matrix $\{R_{ij} = 0,\ i + j < m\}$. Therefore, the first m columns of matrix V are equivalent to the matrix Z, the other to matrix S: $V = ZS$: and:

$$A(\bar{d})S\bar{d}_s = -\bar{T}. \tag{27a}$$

If matrix $A\left(\tilde{\bar{d}}\right)$ has a rank M, matrix $A(\bar{d})S$ should contain M non-degenerate rows and $\bar{d}_S$ is unambiguously defined from a linear equation system as in equation (27). Vector d_z is determined from solving a set of linear equations:

$$Z^T HZ\bar{d}_z = -Z^T\left(\bar{p} + HS\bar{d}_s\right) \tag{28}$$

and, the Lagrangian multipliers are determined as:

$$H\Delta\bar{d} + \bar{p} = A^T(\bar{d})\bar{\psi} \tag{29}$$

and having computed $\Delta\bar{d}$ a new iterative value of the unknown vector $\tilde{\bar{d}}^*$ is calculated as:

$$\tilde{\bar{d}} = \bar{d} + \gamma\,\Delta\bar{d} \tag{30}$$

where a non-negative scalar γ is the step length, minimized Lagrangian function as in equation (16) increment in direction $\Delta\bar{d}$ (Gill *et al.*, 1981).

Having completed computations of the next approximation, the condition of iterative process halt is checked:

$$\|\tilde{\bar{d}} - \bar{d}\|_{R^{N_p}} \le \varepsilon, \tag{31}$$

where ε is a given positive constant. If the condition equation (31) is not satisfied, a new Hessian approximation is defined as a rank-two modification of $H(\bar{d}^*)$:

$$H\left(\tilde{\bar{d}}^*\right) = H(\bar{d}) - \frac{1}{\Delta\bar{d}^T H(\bar{d})\Delta\bar{d}} H(\bar{d})\Delta\bar{d}\Delta\bar{d}^T H(\bar{d}) + \frac{1}{\bar{y}^T\Delta\bar{d}}\bar{y}\bar{y}^T \tag{32}$$

where:

$$\bar{y} = \bar{p}(\bar{d}^*) - A(\bar{d}^*)^T\bar{\Psi}^* - \bar{p}(\bar{d})^T + \bar{\Psi}$$

It remains to consider the Jacobian matrix of constraints $A(\bar{d})$ calculating:

$$A(\bar{d}) = \{A_{(mnj)ik}\} = \left\{\frac{\partial T(R_m, \theta_n, \tau_j)}{\partial d_{ik}}\right\} = \{\vartheta_{(mnj)ik}\} \tag{33}$$

Unfortunately, parameters d_{ik} are included in equations (4-9) implicitly. To obtain the $\vartheta_{(\mathrm{mnj})ik}$ $m = 1, 2, \ldots m,$ $i = 1, 2, \ldots, N_k,$ $k = 1, 2, \ldots, N$ it is reasonable to make a Landau's transformation of variables, which provide the explicit form of equations (4-9) relatively to d_{ik}:

$$r' = (r - R_{l-1})/\Delta_l, r' \in (0,1), r \in (R_{l-1}, R_l) \tag{34}$$

or:

$$r = r' \Delta_l + R_{l-1}$$

Then, equations (4)-(9) can be rewritten as:

$$\left(r'^2 \Delta_l^4 + 2r' \Delta_l^3 R_{l-1} + R_{l-1}^2 \Delta_l^2\right) C_l(T) \frac{\partial T_l}{\partial \tau} = \frac{\partial}{\partial r'}\left(\left(r' \Delta_l + R_{l-1}\right)^2 \lambda(T) \frac{\partial T_l}{\partial r'}\right)$$
$$+ \frac{\Delta_l^2}{\sin\theta} \frac{\partial}{\partial \theta}\left(\lambda(T) \sin\theta \frac{\partial T_l}{\partial \theta}\right), \ r' \in (0, 1), \quad \theta \in (0, \theta_0), \ l = 1, 2, \ \ldots, L,$$
$$\tau \in (\tau_{\min}, \tau_{\max}] \tag{35}$$

$$T_l(r', \theta, \tau_{\min}) = T_{0l}, \quad l = 1, 2, \ \ldots, L \tag{36}$$

$$-\beta_1 \lambda_1(T_1(0, \theta . \tau)) \frac{\partial T_1(0, \theta, \tau)}{\partial r'} + \Delta_1 \alpha_1 T_1(0, \theta, \tau) = \Delta_1 q_1(\theta, \tau) \tag{37}$$

$$-\beta_2 \lambda_L(T_L(1, \theta, \tau)) \frac{\partial T_L(1, \theta, \tau)}{\partial r'} + \Delta_L \alpha_2 T_L(1, \theta, \tau) = \Delta_L q_2(\theta, \tau), \tag{38}$$

$$\lambda_l(T_l(1, \theta, \tau)) \Delta_{l+1} \frac{\partial T_l(1, \theta, \tau)}{\partial r'} = \lambda_{l+1}(T_{l+1}(0, \theta, \tau)) \Delta_l \frac{\partial T_{l+1}(0, \theta, \tau)}{\partial r'},$$
$$l = 1, 2, \ \ldots, L-1, \quad \tau \in (\tau_{\min}, \tau_{\max}], \tag{39}$$

$$-\lambda_l(T_l(1, \theta, \tau)) A_l(T_l(R_l, \theta, \tau)) \frac{\partial T(1, \theta, \tau)_l}{\partial r'} = \Delta_l T_l(1, \theta, \tau) - \Delta_l T_{l+1}(0, \theta, \tau), \tag{40}$$

It can be shown that ${}^{\partial T}/_{\partial d_{l_k}}\left(R'_m, \theta_n, \tau_j\right) = \vartheta_k\left(R'_m, \theta_n, \tau_j\right)$, $R'_m = (R_m - R_{l-1})/\Delta_l$, where $\vartheta_k(r', \theta, \tau)$ satisfies the following set of equation (5):

$$\left(r'^2\Delta_l^4+2r'\Delta_l^3R_{l-1}+R_{l-1}^2\Delta_l^2\right)C_l(T)\frac{\partial\vartheta_{lik}}{\partial\tau}=(r\Delta_l+R_{l-1})^2\lambda\frac{\partial^2\vartheta_{lik}}{\partial r^2}$$

$$+\Delta_l^2\lambda(T)\frac{\partial^2\vartheta_{lik}}{\partial\theta^2}+\left(2(r\Delta_l+R_{l-1})^2\frac{\partial T_l}{\partial r}\frac{d\lambda_l}{dT}+2(r\Delta_l+R_{l-1})\Delta_l\lambda\right)\frac{\partial\vartheta_{lik}}{\partial r}$$

$$+\frac{\Delta_l^2}{\sin\theta}\left(\sin\theta\frac{\partial T_l}{\partial\theta}\frac{d\lambda_l}{dT}+\cos\theta\,\lambda\right)\frac{\partial\vartheta_{lik}}{\partial\theta}+\left((r\Delta_l+R_{l-1})^2\frac{\partial^2T_l}{\partial r^2}\frac{d\lambda_l}{dT}\right.$$

$$+(r\Delta_l+R_{l-1})^2\frac{d^2\lambda_l}{dT^2}\left(\frac{\partial T_l}{\partial r}\right)^2+2(r\Delta_l+R_{l-1})\Delta_l\frac{\partial T_l}{\partial r}\frac{d\lambda_l}{dT}+\Delta_l^2ctg\theta\frac{\partial T_l}{\partial\theta}\frac{d\lambda_l}{dT}$$

$$\left.+\Delta_l^2\frac{d^2\lambda_l}{dT^2}\left(\frac{\partial T_l}{\partial\theta}\right)^2+\Delta_l^2\frac{\partial^2T_l}{\partial r^2}\frac{d\lambda_l}{dT}-\left(r'^2\Delta_l^4+2r'\Delta_l^3R_{l-1}+R_{l-1}^2\Delta_l^2\right)\frac{\partial T_l}{\partial\tau}\frac{dC_l}{dT}\right)\vartheta_{lik}$$

$$+\,{}^{kl}_{\delta}\left(-\left(r'^2 4\Delta_l^3+6r'\Delta_l^2R_{l-1}+2R_{l-1}^2\Delta_l\right)C_l\frac{\partial T_l}{\partial\tau}+4r'\Delta_l\lambda_l\frac{\partial T_l}{\partial r}\right.$$

$$+2(r\Delta_l+R_{l-1})\lambda_l\frac{\partial^2T_l}{\partial r^2}+2(r\Delta_l+R_{l-1})\frac{d\lambda_l}{dT}\left(\frac{\partial T_l}{\partial r}\right)^2$$

$$\left.+2\Delta_l\left(ctg\theta\,\lambda_l\frac{\partial T_l}{\partial r}+\frac{d\lambda_l}{dT}\left(\frac{\partial T_l}{\partial\theta}\right)^2+\lambda_l\frac{\partial^2T_l}{\partial\theta^2}\right)\right)\varphi_{ik}(\theta),$$

$$r'\in(0,1),l=1,2,\ldots,L,\tau\in(\tau_{\min},\tau_{\max}]. \tag{41}$$

$$-\beta_1\lambda_1(T_1(0,\theta,\tau))\frac{\partial\vartheta_{1ik}(0,\theta,\tau)}{\partial r}-\beta_1\frac{\partial T_1}{\partial R}\frac{d\lambda_1}{dT}\vartheta_{1ik}(0,\theta,\tau)$$

$$+\Delta_l\alpha_1\vartheta_{1ik}(0,\theta,\tau)=\delta_k^1\left(q_1(\theta,\tau)-\alpha_1T_1(0,\theta,\tau)\right)\varphi_{ik}(\theta), \tag{42}$$

$$-\beta_2\lambda_L(T_L(1,\theta,\tau))\frac{\partial\vartheta_{Lik}(1,\theta,\tau)}{\partial r}-\beta_2\frac{\partial T_L}{\partial r}\frac{d\lambda_L}{dT}\vartheta_{Lik}(1,\theta,\tau)$$

$$+\Delta_L\alpha_2\vartheta_{Lik}(1,\theta,\tau)=\delta_k^L\left(q_2(\theta,\tau)-\alpha_2T_L(1,\theta,\tau)\right)\phi_{ik}(\theta), \tag{43}$$

$$\Delta_{l+1}\lambda_l(T_l(1,\theta,\tau))\frac{\partial\vartheta_{lik}(1,\theta,\tau)}{\partial r}+\Delta_{l+1}\frac{\partial T_l}{\partial r}r\frac{d\lambda_l}{dT}\vartheta_{lik}(1,\theta,\tau)+\delta_k^{l+1}\lambda_l\frac{\partial T_{l+1}}{\partial r}\varphi_{ik}(\theta)$$

$$=\Delta_l\lambda_{l+1}(T_{l+1}(0,\theta,\tau))\frac{\partial\vartheta_{l+1,ik}(0,\theta,\tau)}{\partial r}+\Delta_l\frac{\partial T_{l+1}}{\partial r}\frac{d\lambda_{l+1}}{dT}\vartheta_{l+1,ik}(0,\theta,\tau)$$

$$+\delta_k^l\lambda_{l+1}\frac{\partial T_{l+1}}{\partial r}\varphi_{ik}(\theta),l=1,2,\ldots,L-1,\quad\tau\in(\tau_{\min},\tau_{\max}), \tag{44}$$

$$
\begin{aligned}
&-\lambda_l \frac{\partial \vartheta_{lik}}{\partial r}(1,\theta,\tau)R_l - \frac{d\lambda_l}{dT}\frac{\partial T_l}{\partial r}R_l\vartheta_{lik}(1,\theta,\tau) - \lambda_l \frac{dR_l}{dT}\frac{\partial T_l}{\partial r}\vartheta_{lik}(1,\theta,\tau) \\
&= \delta_l^k(T_l(1,\theta,\tau) - T_{l+1}(0,\theta,\tau)) + d_l\left(\vartheta_{lik}(1,\theta,\tau) - \vartheta_{l+1,ik}(0,\theta,\tau)\right), \\
&l = 1,2,\ldots,\mathrm{L}-1, \quad \tau \in (\tau_{\min},\ \tau_{\max}), \qquad (45)
\end{aligned}
$$

where δ_k^l – Kronecher's symbol

$$
\delta_k^l = \begin{cases} 1, & if \ \ l = k, \\ 0, & if \ \ l \neq k. \end{cases}
$$

Therefore, the algorithm of search of optimal thickness of multi-layer thermal insulation is complete.

Design of multi-layer thermal protection of variable thickness

As an example of applying the suggested algorithm, let us consider a problem of thickness sampling for a multilayer thermal protection shield of advanced solar probe. Considered problem is just a testing example, which approved the workability of developed algorithm. For this purpose, it is not very important to consider the real technical problems with all external factors, etc.

It is suggested that the spacecraft is three-axis stabilized and pointed with "X" axis toward the Sun; the minimum Sun-spacecraft distance corresponds to 9.74 solar radii. During perihelion passes, the spacecraft will be subjected to the solar flux, ranging between 20,000 and 670,000 W/m^2, as shown in Figure 1. The back surface of the heat shield was assumed thermally insulated (it is traditional majority estimate for design problems).

As a designed structure, a two-layer thermal protection coating with the geometry, presented in Figure 2, is considered. The system includes:

- a carbon-carbon composite material coated with SiC (C-C) and highly porous ceramic material (TZMK-10);
- a carbon-carbon composite material and reticulated vitreous carbon foam (RVC);
- highly porous ceramic material and reticulated vitreous carbon foam; and
- the carbon-carbon composite material and highly porous SiC foam (SiC).

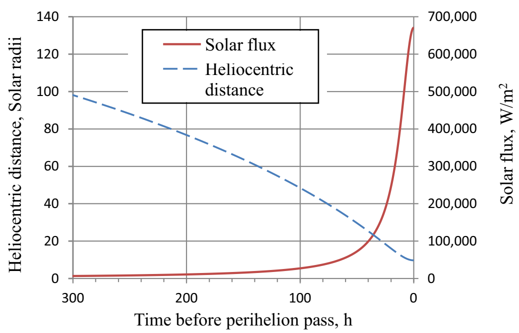


Figure 1. Time dependence of heliocentric distance and solar flux to the front surface of solar probe heat shield

Values of thermal properties of materials used in simulation are given in Table I. The thicknesses of both layers are determined (N = 2) taking into account the temperature constraints between the layers and at the back surface of the system (M = 2). The number of steps by time equals 1,000. The numbers of parameters of approximation are equal to Nk = 5. With this set of numerical parameters, the considered problem is well-posed, and, therefore, it does not need any kind of regularization. In general case, when coefficients of mathematical model (thermal conductivity and heat capacity) are continuous functions, the minimized functional has convex shape, and corresponding Hessian never becomes close to singular, and thickness optimization problem also does not need any kind of regularization. Cases of very specific thermal properties of materials were not considered in this paper.

The temperature constraints on the back surface of the coating and at the boundary of layers were limited to the values of 320 and 900 K, respectively. The results of variable thickness determining for the two-layer thermal protection coating of the solar probe heat shield, having spherical segment shape, are presented in Figure 3. Optimal layer thickness was obtained for four different layer compositions listed in Table II.

The temperatures as a function of time for a few discrete points at the surface are presented in Figure 4. For considered heat flux, just the final temperature constraint became active for any points and any layers. The reason of this fact is the monotonically increasing heat flux at considered time. The effectiveness of considered approach is proved by the simultaneous maximum values (equal to constraint) of all points and all layers.

Through increasing the radius of curvature of a spherical segment heat shield surface, the temperature of the first thermal insulation layer could be greatly reduced, thereby minimizing the adsorbed solar flux and potentially making the shield more effective. However, it should be noted that the shape selection for the sunshield is driven also by the geometrical size of the spacecraft, dimensions of the payload fairing and limitations related

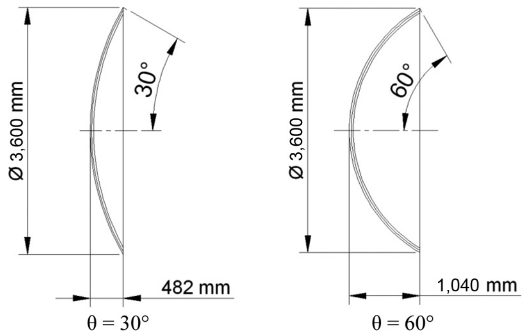
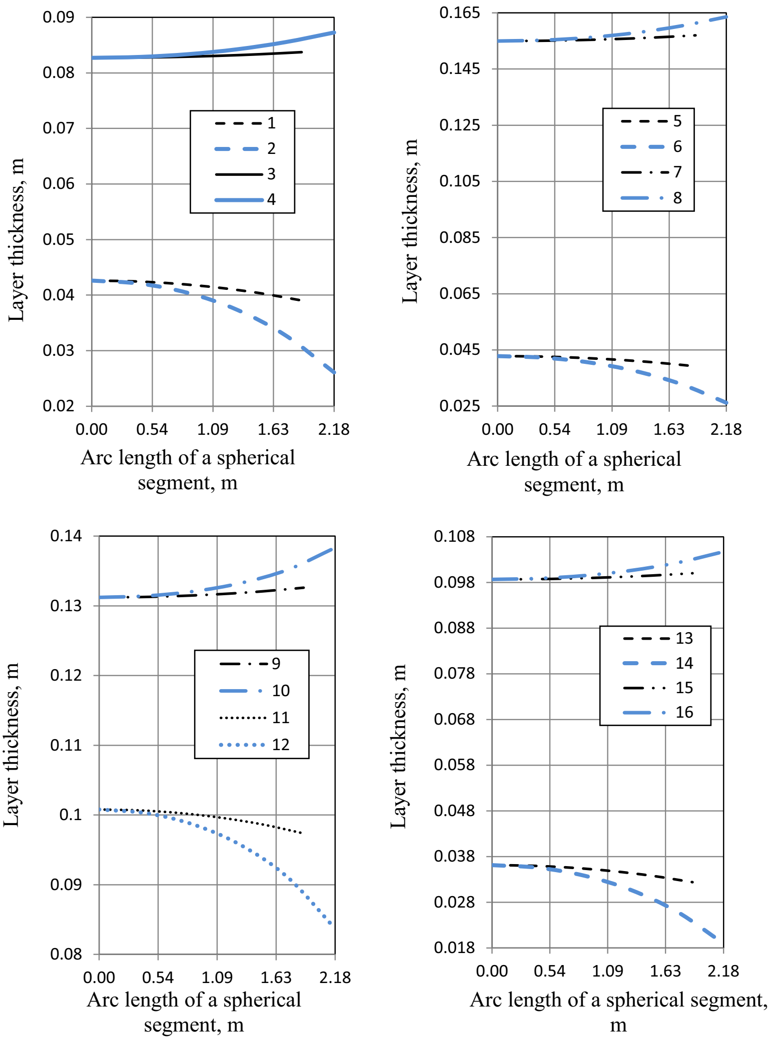


Figure 2.
Geometric characteristics of the solar probe heat shield

Table I.
Thermal properties of materials

Material	λ, (W/(m·K)	C, J/(m^3·K)	ρ, (kg/m^3)
C-C	4.0	2,721,000	1,802
TZMK-10	0.35	184,600	142
RVC	0.6	90,000	57
SiC	2.0	780,000	567

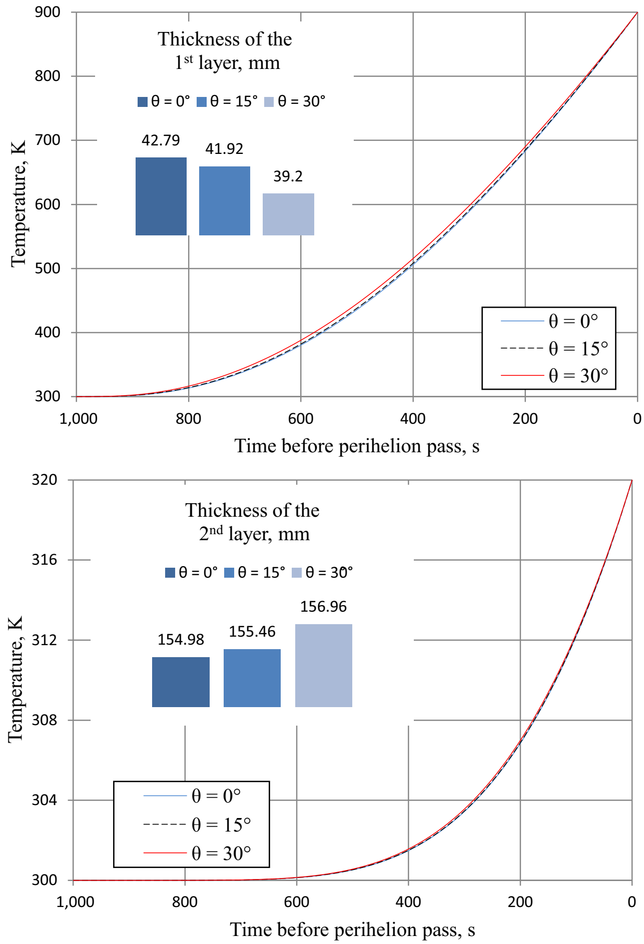


Note: The lower curves correspond to the first layer

Figure 3. Optimal layer thickness of the spherical heat shield for four different layer compositions (according to Table II)

Material of First layer	Curve in Figure 3 θ = 30°	θ = 60°	Material of Second layer	Curve in Figure 3 θ = 30°	θ = 60°
C-C	1	2	TZMK-10	3	4
C-C	5	6	RVC	7	8
TZMK-10	11	12	RVC	9	10
C-C	13	14	SiC foam	15	16

Table II. Curves numbers depending on the layers' composition and heat shield configuration



Note: Materials of the first and second layers are carbon-carbon composite material and reticulated vitreous carbon foam, respectively

Figure 4.
Time dependence of temperature at the boundary of layers and on the back surface of the coating for different angles

to the required fields of view for the remote sensing instruments. A flat surface of the shield would greatly simplify the interface to the payload and ensure a clear view through the shield for the remote sensing instruments.

Conclusion

The procedure proposed in the paper to solve a design problem is based on the method of quadratic approximation of the initial problem statement as a Lagrange formulation. This has allowed to construct a rather universal algorithm applicable without modification for solving a wide range of thermal design problems.

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